Anomalous Hall and Nernst effects in Dirac materials

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Talk is based on: V.P. Gusynin, S.G. Sh., and A.A.Varlamov, PRB **90**, 155107 (14); Low Temp. Phys. **41**, 342 (15).
V.Yu.Tsaran, S.G.Sh., PRB **93**, 075430 (16).

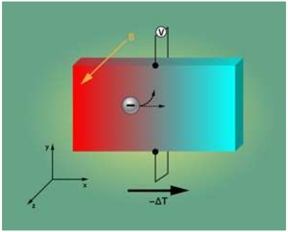
4 December, 2018

Outline

Nernst - Ettingshausen effect (1886)

Walther Nernst
1864 - 1941

Nobel Prize in chemistry (1920) in recognition of his work in thermochemistry. Third law of thermodynamics.



Nernst signal

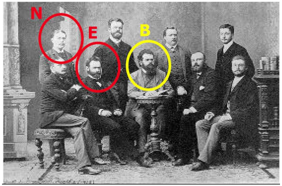
$$e_y(T) = -\frac{\sigma_{xx}\beta_{yx} - \sigma_{yx}\beta_{xx}}{\sigma_{xx}^2 + \sigma_{xy}^2}$$

$$e_y(T) \approx \frac{\beta_{xy}}{\sigma_{xx}}$$

for $\sigma_{xx} \gg |\sigma_{xy}|$

β_{xy} is the thermoelectric coefficient

Sensitive to the details of the electronic structure



Albert von Ettingshausen
1850 - 1932

$e_y = -\frac{E_y}{\nabla_x T} \left[\frac{\mu V}{K}\right]$

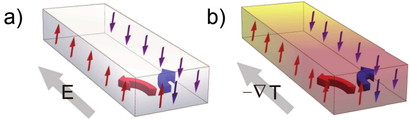
Energy scale:

$k_B/e \sim 86\, \mu V/K$

There are: 1st NE effect: $\nabla_x T \to E_y$

2nd NE effect: $j_x \to \nabla_y T$

SH and SN effects



For NE an external magnetic field $\mathbf{B} \parallel \hat{z} \neq 0$ is required!

Now $\mathbf{B} = 0$, but there is the internal magnetic field or spin-orbit interaction.

SN effect: $\mathbf{j}^s = -\widehat{\beta}^s \boldsymbol{\nabla} T$

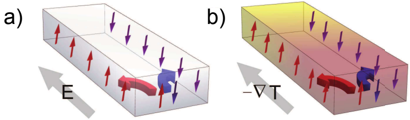
with the thermo-spin tensor, $\widehat{\beta}^s$

Purpose is to study SN effect in low-buckled Dirac materials.

Spin Nernst (SN) effect

Spin caloritronics phenomena

SH and SN effects



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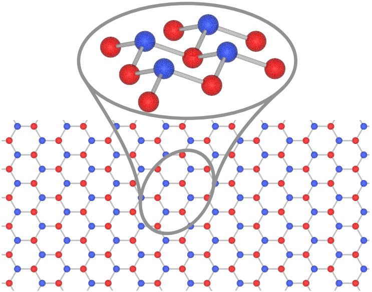
Spin current subtlety

There is no conservation of spin! $\frac{\partial S_z}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{J}_s = \mathcal{T}_z$, where the spin torgue density $\mathcal{T}_z(\mathbf{r}) = \Re e\, \Psi^\dagger(\mathbf{r}) \hat{\tau} \Psi(\mathbf{r})$ with $\hat{\tau} \equiv \frac{d\hat{S}_z}{dt} = \frac{1}{i\hbar}[\hat{S}_z, \hat{H}]$.

When $[\hat{S}_z, \hat{H}] = 0$ the spin torque term is zero and the spin current $\mathbf{J}_s(\mathbf{r}) = \Re e\, \Psi^\dagger(\mathbf{r}) \frac{1}{2} \left\{ \hat{\mathbf{v}}, \hat{S}_z \right\} \Psi(\mathbf{r})$ with the spinor $\Psi^T = (\psi_\uparrow, \psi_\downarrow)$.

J. Shi, P. Zhang, Di Xiao, Q. Niu, PRL **96**, 076604 (06); P. Zhang, Z. Wang, J. Shi, Di Xiao, and Q. Niu, PRB **77**, 075304 (08).

Low-buckled Dirac materials



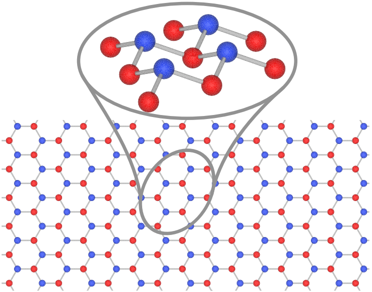
Silicene: vertical distance between sublattices $2d \approx 0.46$Å.
Lattice constant $a = 3.87$Å.
So far grown on Ag and ZrB_2 substrates which are both conductive – no transport measurements as yet.
2D sheets of Ge, Sn, P and Pb atoms (the materials germanene, stanene and phosphorene).

Strong intrinsic spin-orbit interaction in contrast to graphene

$$H_{\mathrm{SO}} = i\frac{\Delta_{\mathrm{SO}}}{3\sqrt{3}} \sum_{\substack{\langle\langle i,j\rangle\rangle \\ \sigma\sigma'}} c^{\dagger}_{i\sigma}(\boldsymbol{\nu}_{ij}\cdot\boldsymbol{\sigma})_{\sigma\sigma'} c_{j\sigma'}$$

with $\Delta_{\mathrm{SO}} \sim 10\,\mathrm{meV}$, $\nu^z_{ij} = \pm 1$.

Low-buckled Dirac materials



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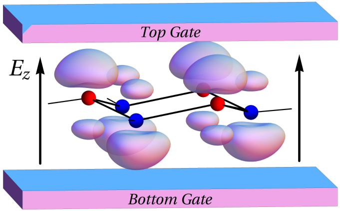

Perpendicular to the plane electric field E_z opens the tunable gap $\Delta_z = E_z d$.
Interplay of two gaps: Δ_{SO} and Δ_z.

Low-energy Hamiltonians and main goals

1. Toy model: two-component Dirac fermions model

$$\mathcal{H} = \hbar v_F(k_x\tau_1 + k_y\tau_2) + \Delta\tau_3 - \mu\tau_0.$$

The mass Δ breaks TR symmetry. To study off-diagonal part of the TE tensor $\widehat{\beta}$:

$$\mathbf{j} = \widehat{\sigma}\mathbf{E} - \widehat{\beta}\nabla T$$

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$$\mathbf{j} = \widehat{\sigma}\mathbf{E} - \widehat{\beta}\nabla T$$

2. Silicene

$$\mathcal{H}_\eta = \sigma_0 \otimes [\hbar v_F (\eta k_x \tau_1 + k_y \tau_2) + \Delta_z \tau_3 - \mu \tau_0] - \eta \Delta_{\mathrm{SO}} \sigma_3 \otimes \tau_3,$$

$\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$ – sublattice and spin; $\mathbf{k}$ is measured from the $\mathbf{K}_\eta$ points.
There is a spin $\sigma = \pm$, and valley $\eta = \pm$ dependent gap $\Delta_{\eta\sigma} = \Delta_z - \eta\sigma\Delta_{\mathsf{SO}}$ or mass $\Delta_{\eta\sigma}/v_F^2$, where v_F is the Fermi velocity.
When $\Delta_{\eta\sigma} = 0$ come back to graphene.
TRS is unbroken for any $\Delta_{\eta\sigma}$.
To study off-diagonal part of the thermo-spin tensor $\widehat{\beta}^s$:

$$\mathbf{j}^s = \widehat{\sigma}^{sc}\mathbf{E} - \widehat{\beta}^s \nabla T$$

Anomalous Hall effect in the toy model

For $B = 0$ equation of motion for $\eta = +$

$$\dot{\mathbf{v}} = \frac{1}{i\hbar}[\mathbf{v}, H] = 2v_F^2 \mathbf{k} \times \boldsymbol{\tau} - \frac{2\Delta}{\hbar} \mathbf{v} \times \mathbf{e}_z, \quad \mathbf{v} = v_F \boldsymbol{\tau}.$$

Here the first term corresponds to Zitterbewegung and the second term corresponds to the Lorentz force due to magnetic field $\mathbf{B}_{eff} \perp$ plane, where $B_{eff} \propto \Delta$. This is related to the Haldane model, Phys. Rev. Lett. **61**, 2015 (1988), also T. Ando, J. Phys. Soc. Jpn. **84**, 114705 (15).
For $T = 0$ the intrinsic (not induced by disorder) AHE

$$\sigma_{xy}^{\eta} = -\frac{e^2 \mathrm{sgn}\,(\eta\Delta)}{4\pi\hbar} \begin{cases} 1, & |\mu| \le |\Delta|, \\ |\Delta|/|\mu|, & |\mu| > |\Delta|. \end{cases}$$

For $|\mu| > |\Delta|$ the vertex corrections modify the result N.A. Sinitsyn, J.E. Hill, H. Min, J. Sinova, and A.H. MacDonald, PRL **97** 106804 (06). Moreover, the standard diagrammatic approach fails A. Ado, I.A. Dmitriev, P.M. Ostrovsky, and M. Titov, Europhys. Lett. **111**, 37004 (15).

We will return to the AHE in the second part of my talk.

SHE scenario for silicene

Silicene for $B = 0$ TR unbroken $\sigma_{xy} = \sum_{\xi,\sigma=\pm} \xi \sigma_{xy}(\Delta \to \Delta_{\xi\sigma}) = 0.$

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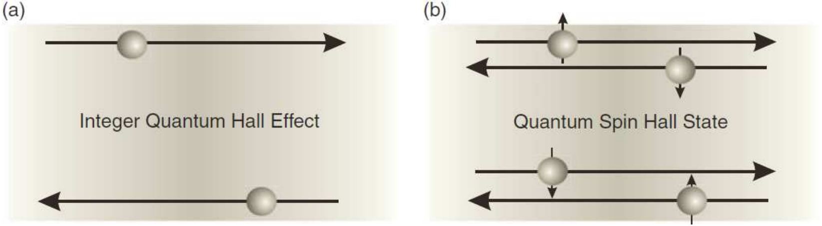
Kane-Mele scenario of SHE. It occurs due to the presence of two subsystems with $\sigma = \pm$ exhibiting the quantum Hall effect: $\sigma_{xy}^{S_z} = -\frac{\hbar}{2e}\sum_{\xi,\sigma=\pm} \xi\sigma\sigma_{xy}(\Delta \to \Delta_{\xi\sigma})$.



Proposed for graphene in C.L. Kane and E.J. Mele, PRL **95**, 226801 (05). For $\Delta_z = 0$
$\sigma_{xy}^{S_z} = -\frac{e}{2\pi}\,\mathrm{sgn}\,(\Delta_{\mathrm{SO}})\left[\theta(|\Delta_{\mathrm{SO}}| - |\mu|) + \frac{|\Delta_{\mathrm{SO}}|}{|\mu|}\theta(|\mu| - |\Delta_{\mathrm{SO}}|)\right]$ For $|\mu| < |\Delta_{\mathrm{SO}}|$ – quantum spin Hall insulator.
$\sigma_{xy}^{S_z}$ is measured in the units of $e/(4\pi)$.

Why an interesting physics can be expected?

Mott relation for thermoelectric coefficient (toy model):

$$\beta_{xy} = -\frac{\pi^2 k_B^2}{3e} T \frac{\partial \sigma_{xy}(\mu, \Delta, T=0)}{\partial \mu}$$

Then the Nernst signal for $\sigma_{xx} \gg |\sigma_{xy}|$ and $|\mu| > |\Delta|$:

$$e_y(T) \approx \frac{\beta_{xy}}{\sigma_{xx}} = -\left(\frac{k_B}{e}\right) \frac{\pi e^2}{12\hbar \sigma_{xx}} \frac{k_B T \Delta \mathrm{sgn}\,(\mu)}{\mu^2}.$$

The order of magnitude is $e_y(T) \sim k_B/e \sim 86\,\mu V/K$.

Tuning the position of μ by changing the gate voltage one gains from 3 to 4 orders of magnitude in e_y as compared to the normal nonmagnetic metals, where $e_y \sim 10\,nV/K$ per Tesla.

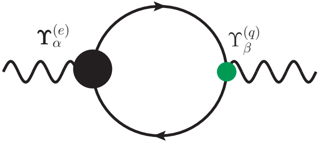
No AHE in silicene, but it should be SHE and large spin Nernst effect!

Problem with the Kubo formula: toy model

Consider the usual definition of the thermolectric tensor

$$\tilde{\beta}_{xy} = -\frac{\hbar}{T}\lim_{\omega\to 0}\frac{Q_{xy}^{eq(R)}(\omega)}{\omega},$$

where $Q_{xy}^{eq(R)}$ is the retarded response function of the electric and heat currents.



$\boldsymbol{\Upsilon}_\alpha^{(e)}$ – electric current vertex; $\Upsilon_\alpha^{(q)}$ – heat current vertex.
In the clean case (bare bubble) and in the limit $T \to 0$

$$\tilde{\beta}_{xy} = -\frac{e}{4\pi\hbar T}[\Delta \mathsf{sgn}\,(\mu)\theta(|\mu| - |\Delta|) + \mu \mathsf{sgn}\,(\Delta)\theta(|\Delta| - |\mu|)]$$

diverges!
At $T = 0$ the thermoelectric tensor must become zero: it describes the transport of entropy, which, in accordance with the third law of thermodynamics, has to become zero when $T \to 0$.

Modified Kubo formula

Role of the magnetization currents

It was shown by Yu.N. Obraztsov, Fiz. Tverd. Tela **6**, 414 (1964) [see also N. R. Cooper, B. I. Halperin, and I. M. Ruzin, PRB **55**, 2344 (1997); T. Qin, Q. Niu, and J. Shi, PRL 107, 236601 (11)] that in the presence of an effective magnetic field, the off-diagonal thermal transport coefficient $\tilde{\beta}_{xy}$ has to be corrected by including of the magnetization M_z term: so that the correct thermoelectric tensor

$$\beta_{xy} = \tilde{\beta}_{xy} + \frac{cM_z}{T},$$

where (V.P. Gusynin, S.G. Sh., and A.A.Varlamov, PRB **90**, 155107 (14).)

$$M_z(B=0) = \frac{e\,\mathrm{sgn}\,(\eta\Delta)T}{4\pi\hbar c}\left[\ln\cosh\frac{\mu+|\Delta|}{2k_BT} - \ln\cosh\frac{\mu-|\Delta|}{2k_BT}\right].$$

In the limit $T \to 0$ it cancels out the diverging part of $\tilde{\beta}_{xy}$ and the third law of thermodynamics is restored.

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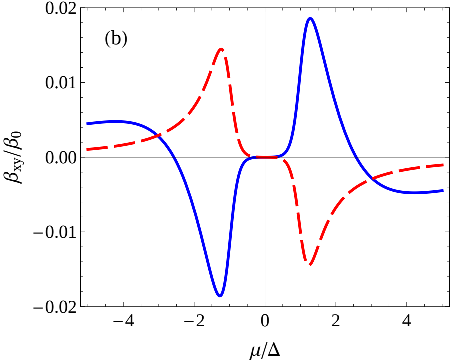
In the limit $T \to 0$ it cancels out the diverging part of $\tilde{\beta}_{xy}$ and the third law of thermodynamics is restored.
For silicene the divergence is compensated by the "spin magnetization" $M_z^{S_z} = -\frac{\hbar}{2e}\sum_{\xi,\sigma=\pm}\xi\sigma M_z(\Delta \to \Delta_{\xi\sigma})$, which is nonzero even when the TR symmetry is unbroken. The orbital magnetization $M_z = \sum_{\xi,\sigma=\pm}\xi M_z(\Delta \to \Delta_{\xi\sigma}) = 0$.

Thermo-electric and -spin coefficients: results

Toy model

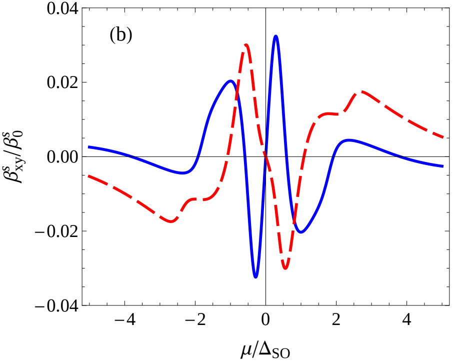


Thermoelectric coefficient $\beta_{xy}(\mu)$ in units of $\beta_0 = k_B e/\hbar$.

Red line — bubble approximation

Blue line — dressed vertex

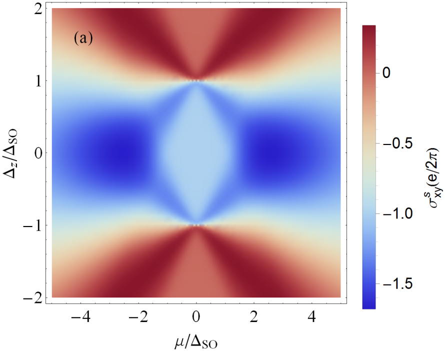
Spin NE silicene

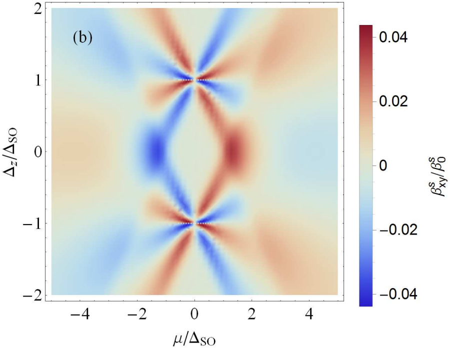


Thermospin coefficient $\beta^{S_z}_{xy}(\mu)$ in units of $\beta^s_0 = k_B/2$.

Crossing $\beta_{xy}(\mu \neq 0) = 0$ is caused by nonmonotonic dependence $\sigma_{xy}(\mu) = 0$ related to the vertex. Other diagrams modify this result.

Results: electric vertex included





Spin Hall conductivity $\sigma_{xy}^{S_z}(\mu, \Delta_z)$ in units of $\sigma_0^s = e/(2\pi)$

Thermo-spin coefficient $\beta_{xy}^{S_z}(\mu, \Delta_z)$ in units of $\beta_0^s = k_B/2$

as functions of the chemical potential μ and the sublattice asymmetry gap Δ_z in the units of $\Delta_{SO} > 0$.

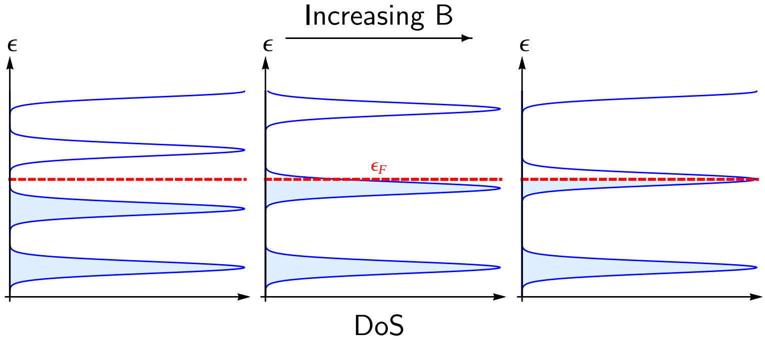
Summary for the Nernst effect part

- We illustrated how the standard Kubo formula has to be altered by the magnetization term leading to the correct answer off-diagonal thermoelectric coefficient. Note that $B = 0$! The whole story with the corrections of the Kubo formula is rather counter-intuitive.
- The bare bubble case was presented only for illustration. A more formal generic proof of the cancellation of the diverging terms was also made.
- Spin Nernst effect is strong, so potentially may be observable.
- A possibility to distinguish different cases with monotonic and nonmonotonic dependence $\sigma_{xy}(\mu)$ and $\sigma_{xy}^{S_z}(\mu, \Delta_z)$ due to the vertex and other diagrams.

Now we pass to the anomalous Hall effect and oscillations of the anomalous Hall conductivity.

Origin of magnetic oscillations

Density of states (DoS) is $D(\epsilon) = \frac{|eB|}{2\pi\hbar c} \sum_n \delta(\epsilon - \epsilon_n)$.



Oscillations of the magnetization were predicted in the paper on Landau diamagnetism, Z. Phys. **64**, 629 (1930), although Landau levels were originally discovered by I. Rabi, Z. Phys. **49**, 507 (1928) for the Dirac equation! Only then Schrödinger equation was considered by M.P. Bronstein and I.A. Frenkel, "Quantizing of free electorns in magnetic field", J. Russian Phys. and Chem. Soc. **62**, 485 (1930).

Oscillations of density of states

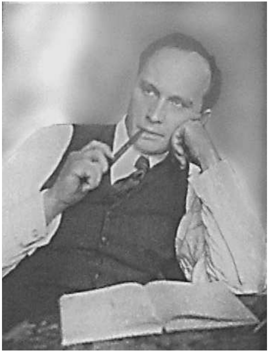
Assumption: Landau levels have a Lorentzian shape $\delta(\epsilon) \to \frac{\Gamma}{\pi(\epsilon^2+\Gamma^2)}$, with the impurity scattering rate $\Gamma = \hbar/(2\tau)$ independent of energy or magnetic field, and τ being a mean free time of quasiparticles.
In perpendicular magnetic field the DoS acquires an oscillatory component $\tilde{D}(\mu, B)$,

$$\frac{\tilde{D}(\mu, B)}{D_0(\mu)} = 2\sum_{s=0}^{\infty} \cos\left[2\pi s\left(\frac{cS(\mu)}{2\pi e\hbar B} + \frac{1}{2} + \beta\right)\right] R_T(s\lambda) R_D(s),$$

where $D_0(\mu)$ is DoS in the absence of magnetic field.
Here $S(\mu)$ is the electron orbit area in the momentum space, μ is the chemical potential and β is the topological part of the Berry phase,
$R_T(s\lambda) = \frac{s\lambda}{\sinh s\lambda}$ with $\lambda = \frac{2\pi^2 k_B T}{\hbar\omega_c}$ is *the temperature amplitude factor*,
$R_D(s) = \exp\left(-\frac{\pi s}{\omega_c \tau}\right)$ is *the Dingle factor*.

Shubnikov–de Haas (SdH) oscillations

Lev Shubnikov



$1901 - 1937$

Worked in the Leiden cryogenic laboratory of Wander Johannes de Haas from 1926 – 1930. He established the first Soviet cryogenic laboratory in Kharkiv, Ukraine.

The SdH oscillations are described theoretically by

$$\sigma_{xx}(B,\mu) = \frac{\sigma_0}{1+\omega_c^2\tau^2}\left[1+\gamma f(\omega_c\tau)\frac{\tilde{D}(\mu,B)}{D_0(\mu)}\right],$$

where $\sigma_0 = e^2|n^*|\tau/m^*$ is the conductivity for $B=0$.

For the Dirac case the carrier imbalance is $|n^*| = (\mu^2-\Delta^2)/(4\pi\hbar^2 v_F^2)$. Here $f(\omega_c\tau)$ is a smooth function of $\omega_c\tau$.

Under the assumption of constant Γ in the bare bubble approximation, the function $f(\omega_c\tau)=1$. $\gamma=2$ for 2D electron gas and $\gamma=1$ for the Dirac fermions.

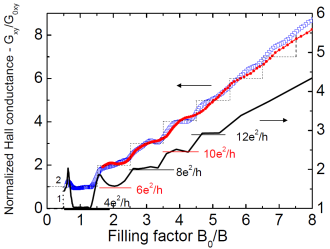
T. Ando, J. Phys. Soc. Jpn. **37**, 1233 (1974); A. Isihara and L. Smrčka, J. Phys. C **19**, 6777 (1986); V.P. Gusynin and S.G.Sh, PRB 71, 125124 (05) for DF.

SdH oscillations of the Hall conductivity

The Hall conductivity is not just a monotonic or steplike function of μ and/or B, towards the quantum Hall regime, but also contains the oscillatory part:

$$\sigma_{xy}(B,\mu) = -\frac{\sigma_0 \omega_c \tau \,\mathrm{sgn}\,(eB)\,\mathrm{sgn}\,\mu}{1+\omega_c^2\tau^2}\left[1 - \frac{g(\omega_c\tau)}{\omega_c^2\tau^2}\frac{\tilde{D}(\mu,B)}{D_0(\mu)}\right],$$

where $g(\omega_c\tau)$ is a smooth function of $\omega_c\tau$.
A. Isihara and L. Smrčka, J. Phys. C **19**, 6777 (1986). Self-consistent consideration of impurities brings $g(\omega_c\tau) = \frac{3\omega_c^2\tau^2}{1+\omega_c^2\tau^2}$.
We obtained that in the bubble approximation for the DF for $\Delta = 0$ and small Γ, the weight of the oscillatory part, $g(\mu\tau)/(\omega_c\tau)^2 = |\mu|/(\omega_c^2\tau\hbar)$.



pre QHE regime in graphite

Question: What is about quantum magnetic oscillations of σ_{xy} for AHE?

Minimal model is the toy model in magnetic field

Two-component Dirac fermions model

$$\mathcal{H}_\eta = v_F \left[\eta\tau_1\left(\hat{p}_x + \frac{e}{c}A_x\right) + \tau_2\left(\hat{p}_y + \frac{e}{c}A_y\right)\right] + \Delta\tau_3 - \mu\tau_0,$$

where the Pauli matrices τ_μ act in the pseudospin space, v_F is the Fermi velocity.
$\eta = \pm$ corresponds to the two independent $\mathbf{K}_\pm$-points.
The fermion doubling [Nielsen and M Ninomiya] guarantees that the ordinary solid-state Dirac materials are always described by "pairs" of the Hamiltonians.
$\mathcal{H}_\eta$ on its own breaks the time-reversal symmetry (TRS).
The magnetic field $\mathbf{B} = \nabla \times \mathbf{A} = B\mathbf{e}_z \perp$ to the plane. The Landau level energies are

$$\epsilon_{n\eta} = \begin{cases} -\eta\Delta \mathrm{sgn}(eB), & n = 0, \\ \pm M_n, & n = 1, 2, \ldots, \end{cases} \qquad M_n = \sqrt{\Delta^2 + 2nv_F^2\hbar|eB|/c}.$$

The $n = 0$ "zero-mode" level breaks both the particle-hole symmetry and invariance under $B \to -B$. The TRS remains broken in the $B = 0$ limit, making possible the quantum Hall effect in the absence of magnetic field.

The oscillatory part of the anomalous Hall conductivity

We find that for $|\mu| > |\Delta|$ the final result is

$$\sigma_{xy}^{\rm AH}(B,\Delta,\mu) = -\frac{e^2}{4\pi\hbar}\frac{\eta\Delta}{|\mu|}\frac{1}{1+\omega_c^2\tau^2}\left[1+\gamma\frac{\tilde{D}(\mu,B)}{D(\mu)}\right]. \qquad (1)$$

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The structure of Eq. (1) resembles the diagonal conductivity

$$\sigma_{xx}(B,\mu) = \frac{\sigma_0}{1+\omega_c^2\tau^2}\left[1+\gamma f(\omega_c\tau)\frac{\tilde{D}(\mu,B)}{D_0(\mu)}\right], \qquad (2)$$

rather than the normal Hall term

$$\sigma_{xy}^{\mathrm{H}}(B,\mu) = -\frac{\sigma_0\omega_c\tau\,\mathsf{sgn}\,(eB)\,\mathsf{sgn}\,\mu}{1+\omega_c^2\tau^2}\left[1-\frac{g(\omega_c\tau)}{\omega_c^2\tau^2}\frac{\tilde{D}(\mu,B)}{D_0(\mu)}\right]. \qquad (3)$$

The oscillating term in Eq. (1) is not damped by the $1/(\omega_c\tau)^2$ factor and has the same weight as the constant term for all strengths of the magnetic field. The absence of the $\omega_c\tau$ prefactor makes possible the observation of the oscillations of the anomalous Hall conductivity even in the low field regime.

A similar to Eq. (1) expression may be expected for any AHE.

Conclusions to Part II

- SdH oscillations of the anomalous Hall conductivity are predicted.
- We [V.Yu.Tsaran, S.G.Sh., PRB **93**, 075430 (16)] have also considered real systems and predicted that SdH oscillations of the anomalous Hall conductivity can be observed in graphene (nonlocal measurements), low-buckled Dirac materials (spin Hall conductivity) and magnetic topological insulators.
- The specific model was considered, but the results should be valid for a wider class of AHE systems.
- The simultaneous measurements of the SdH oscillations in both ρ_{xx} and ρ_{xy} showed the deviations that were attributed to localization. This method, however, did not become widespread because of the difficulties of measuring the SdH oscillations of the Hall resistivity.
  We hope that the situation will change for AHE.

Thank you very much for listening!

General results: symmetry properties

The full Hall conductivity is

$$\sigma_{xy}(B,\Delta,\mu) = \sigma_{xy}^{\mathrm{H}}(B,\Delta,\mu) + \sigma_{xy}^{\mathrm{AH}}(B,\Delta,\mu).$$

Here σ_{xy}^{H} is the usual normal Hall term which is absent for $B=0$

$$\sigma_{xy}^{\mathrm{H}}(B,\Delta,\mu) = -\sigma_{xy}^{\mathrm{H}}(-B,\Delta,\mu) = \sigma_{xy}^{\mathrm{H}}(B,-\Delta,\mu) = -\sigma_{xy}^{\mathrm{H}}(B,\Delta,-\mu)$$

and it changes sign under the reversal of carrier type.
$\sigma_{xy}^{\mathrm{AH}}$ is a new anomalous Hall term

$$\sigma_{xy}^{\mathrm{AH}}(B,\Delta,\mu) = \sigma_{xy}^{\mathrm{AH}}(-B,\Delta,\mu) = -\sigma_{xy}^{\mathrm{AH}}(B,-\Delta,\mu) = \sigma_{xy}^{\mathrm{AH}}(B,\Delta,-\mu).$$

It does not feel neither the sign of B nor the difference between electrons and holes.
Onsager relation: $\sigma_{xy}(B,\Delta,\mu) = \sigma_{yx}(-B,-\Delta,\mu)$, that generalizes a usual relation $\sigma_{xy}^{\mathrm{H}}(B) = \sigma_{yx}^{\mathrm{H}}(-B)$ for the normal Hall conductivity.

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Normally $\sigma_{xy}^{\mathrm{AH}}$ term cancels out after summation over $\eta=\pm$.
Problem is how to observe the AHE in the Dirac materials with two valleys?